

FINITE HOMOLOGICAL DIMENSION AND A DERIVED EQUIVALENCE

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ABSTRACT. For a Cohen-Macaulay ring R , we exhibit the equivalence of the bounded derived categories of certain resolving subcategories, which, amongst other results, yields an equivalence of the bounded derived category of finite length and finite projective dimension modules with the bounded derived category of projective modules with finite length homologies. This yields isomorphisms of K-theory and Witt groups (amongst other invariants) and improves on terms of associated spectral sequences and Gersten complexes.

1. INTRODUCTION

Let R be a commutative Noetherian ring. Let $\mathcal{P}(R)$ be the category of finitely generated projective modules, $\overline{\mathcal{P}}(R)$ be the category of finite projective dimension and \mathcal{M}_{fl} be the category of finite length modules. Let $D_{fl}^b(\mathcal{P}(R))$ be the full subcategory of $D^b(\mathcal{P}(R))$ consisting of bounded complexes with finite length homologies.

The category $\mathcal{M}_{fl} \cap \overline{\mathcal{P}}(R)$ has received much attention in recent years. Bass conjectured that $\mathcal{M}_{fl} \cap \overline{\mathcal{P}}(R)$ contains nonzero modules if and only if the ring is Cohen-Macaulay. This conjecture was resolved in the affirmative by the famed new intersection theorem of [20]. Furthermore, in order to discuss intersection multiplicity over nonregular rings, Roberts and Srinivas show in [19, Proposition 2] that the Grothendieck groups of $\mathcal{M}_{fl} \cap \overline{\mathcal{P}}(R)$ and $D_{fl}^b(\mathcal{P}(R))$ coincide when R is a Cohen-Macaulay ring.

We prove the following theorem which appears as Theorem 4.12.

Theorem 1.1. *If R is a local ring, then R is Cohen-Macaulay if and only if there is an equivalence of derived categories*

$$D^b(\overline{\mathcal{P}}(R) \cap \mathcal{M}_{fl}) \simeq D_{fl}^b(\mathcal{P}(R)).$$

When R is Cohen-Macaulay, the equivalence is a special case of our main result, Theorem 4.7. The main difficulty in proving the equivalence lies in showing that the natural functor $D^b(\overline{\mathcal{P}}(R) \cap \mathcal{M}_{fl}) \rightarrow D_{fl}^b(\mathcal{P}(R))$ is essentially surjective and full, where $D_{fl}^b(\overline{\mathcal{P}}(R))$ is the thick subcategory of $D^b(\overline{\mathcal{P}}(R))$ consisting of complexes with finite length homologies. We overcome this difficulty using a subtle induction on the number of homology modules, an argument which dates back to the preprint [7] (refer to Theorem 3.1 and Theorem 3.3 for our versions).

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When R is not Cohen-Macaulay, the equivalence fails because the new intersection theorem [20] asserts that such a ring R never admits a finite length, finite projective dimension module. Thus $D^b(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{fl}) = 0$. However, the Hopkins-Neeman theorem [11, 15] states that the thick subcategories of $D^b(\mathcal{P}(R))$ are in bijective correspondence with specialization closed subsets of $\text{Spec}(R)$, hence $D_{fl}^b(\overline{\mathcal{P}(R)})$ cannot be 0. See the proof of Theorem 4.12 for more details.

Using Theorem 1.1, when R is Cohen-Macaulay, we can conclude that the non-connective \mathbb{K} -theory spectra of \mathcal{M}_{fl} and $D_{fl}^b(\mathcal{P}(R))$ are homotopy equivalent. Special cases of this result for the connective K -theory spectrum can be obtained from [28, Ex. 5.7], and similar results comparing the K_0 groups are in [19, Proposition 2] (as previously mentioned) and [8]. The homotopy equivalence of the \mathbb{K} -theory spectra is obtained by putting together two equivalences in Theorem 4.5 and Lemma 4.6, both of which are induced through natural functors of the chain complex categories (a zigzag of equivalences induced from chain complex functors).

The equivalence of K groups allows us to refine the terms in weak Gersten complexes for certain non-regular schemes. Let X be a Noetherian scheme, c a non-negative integer, $\text{Coh}(X)$ the category of coherent \mathcal{O}_X -modules, $\text{Coh}(X)^c$ the subcategory of modules with codimension at least c , and $D^c(\text{Coh}(X))$ the subcategory of $D^b(\text{Coh}(X))$ consisting of complexes with homologies in $\text{Coh}(X)^c$. Using the natural coniveau filtration by codimension, one obtains the classical Brown-Gersten-Quillen spectral sequences of \mathbb{K} -groups which abut to the \mathbb{K} -theory of $\text{Coh}(X)$. Applying Quillen's localization and dévissage theorems, the terms occurring in these sequences can be identified with \mathbb{K} -groups of the residue fields of the points. Classically, these spectral sequences were applied in the case when X was Noetherian, regular and separated, in which case they converged to the \mathbb{K} -groups of X [18]. This is because for such X there is a well-known equivalence $D^c(VB_X) \xrightarrow{\xi} D^c(\text{Coh}(X))$ (since there is an ample family of line bundles) which yields $D^b(\text{Coh}(X)^c) \simeq D^c(VB_X)$, where VB_X is the category of locally free sheaves over X . The philosophy thus is that if one can understand the \mathbb{K} -groups for fields and transfer maps between them, one can compute the global \mathbb{K} -groups.

However, without the regularity assumption, ξ is not an equivalence and hence the classical spectral sequences can be used to compute only the coherent \mathbb{K} -groups (better known as G -groups) but not the usual \mathbb{K} -groups. This entire discussion applies for several other generalized cohomology theories, for example triangular Witt groups [6] and Grothendieck-Witt groups [31, 25]. When X is Gorenstein, the corresponding result for coherent Witt groups is in [10].

In [4], niveau and coniveau spectral sequences are established for the usual \mathbb{K} -theory over a (topologically) Noetherian scheme with a bounded dimension function. Similarly weak Gersten complexes are defined. However, the terms occurring in these spectral sequences involve abstract derived categories with support $\mathbb{K}_i(D^b(\mathcal{O}_{X,x} \text{ on } \{x\}))$. Now, when X is Cohen-Macaulay at every stalk, using Theorem 1.2, these terms can be identified with the \mathbb{K} -groups of an actual category of modules, that is, we can rewrite these as $\mathbb{K}_i(\overline{\mathcal{P}(\mathcal{O}_{X,x})} \cap \mathcal{M}_{fl}(\mathcal{O}_{X,x}))$, thus obtaining refined spectral sequences and Gersten complexes in Theorem 5.4. Thus, the philosophy can now be changed to understanding the \mathbb{K} -groups of $\overline{\mathcal{P}(R)} \cap \mathcal{M}_{fl}(R)$ when R is a Cohen-Macaulay local ring, and maps between them.

Furthermore, the equivalence in Theorem 1.1 preserves each categories' natural dualities, which induces isomorphisms of triangular Witt and Grothendieck-Witt

groups, see Theorem 5.5 and Remark 5.7 . Similarly, in Theorem 5.8 we compare a recent definition [13] of triangular Witt groups for certain subcategories of triangulated categories with duality (for example $W^i(D_{\mathcal{P}(R)}^b(\mathcal{P}(R)))$). Indeed, this article grew out of discussions between the authors on such comparisons after a series of talks by Satya Mandal at KU.

Theorem 1.1 is actually the corollary of a much more general statement involving homological dimension. Homological dimension of a module M with respect to a category \mathcal{A} , or the \mathcal{A} -dimension of M , is the smallest resolution of M by modules in \mathcal{A} . A resolving subcategory is in some sense the "correct" type of category over which to take homological dimension. The following is the main theorem of this article which features as Theorem 4.7 :

Theorem 1.2. *Let \mathcal{A} be a resolving subcategory of $\text{mod}(R)$, $\bar{\mathcal{A}}$ the category of modules of finite \mathcal{A} -dimension, and \mathcal{L} a Serre subcategory satisfying condition $(*)$, (see Definition 2.9). Then there is an equivalence of derived categories*

$$D^b(\bar{\mathcal{A}} \cap \mathcal{L}) \simeq D_{\mathcal{L}}^b(\mathcal{A})$$

where $D_{\mathcal{L}}^b(\mathcal{A})$ is the full subcategory of $D^b(\mathcal{A})$ of complexes whose homologies lie in \mathcal{L} .

As with the special case, we can conclude in Theorem 5.1 that the non-connective \mathbb{K} -theory spectra are homotopy equivalent. Also, working in this generality allows us to prove Corollary 4.11 which generalizes an important consequence of the Hopkins-Neeman theorem [11, 15].

A brief word on the organization of the article. In Section 2, we give definitions and preliminaries needed in the article. The proof of Theorem 1.2 uses Theorem 3.1 to reduce the lengths of complexes via a suitable Koszul complex and Theorem 3.3 which makes the reduction functorial and amenable to use in the derived category. Section 3 is devoted to proving Theorem 3.1 and Theorem 3.3. These theorems are crucially used in Section 4, where we prove the main theorems 4.5 and 4.7. In Section 5, we use the main theorems to obtain the natural consequences for \mathbb{K} -theory, derived Witt and Grothendieck-Witt groups. In Section 6, we list several questions and examples of interest.

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2. PRELIMINARIES

We fix a commutative Noetherian ring R throughout the article. Let $\text{mod}(R)$ denote the category of finitely generated R -modules. Let $\mathcal{P}(R)$ denote the full subcategory of projective modules.

Definition 2.1. A commutative Noetherian local ring is said to be Cohen-Macaulay if its depth equals its dimension. A commutative Noetherian ring R is Cohen-Macaulay if each localization $R_{\mathfrak{p}}$ is Cohen-Macaulay local.

When R is Cohen-Macaulay and local, we denote the full subcategory of maximal Cohen-Macaulay modules (i.e. modules M which satisfy $\text{depth}(M) = \dim(M)$) by $MCM(R)$.

PRELIMINARIES : RESOLVING, THICK, AND SERRE SUBCATEGORIES

Resolving subcategories allow one to generalize the notion of projective dimension and Gorenstein dimension. For a subset $\mathcal{C} \subset \text{mod}(R)$ and a module $M \in \text{mod}(R)$, we say that $\dim_{\mathcal{C}} M = n$ if $n \in \mathbb{N}$ is the smallest number such that there is an exact sequence

$$0 \rightarrow C_n \rightarrow \cdots \rightarrow C_0 \rightarrow M \rightarrow 0$$

with $C_0, \dots, C_n \in \mathcal{C}$. For an arbitrary category \mathcal{C} , $\dim_{\mathcal{C}}$ does not have nice properties. However, when \mathcal{C} is resolving, $\dim_{\mathcal{C}}$ behaves similarly to projective and Gorenstein dimension. Resolving subcategories also allow us to simultaneously study usual and coherent theory for invariants such as Grothendieck-Witt groups and \mathbb{K} -theory. They were first studied in [1] and have been of considerable interest recently. We summarize below the definitions, examples and results needed in this article.

Definition 2.2. Given a ring R , a full subcategory $\mathcal{C} \subset \text{mod}(R)$ is resolving if

- (1) R is in \mathcal{C} .
- (2) $M \oplus N$ is in \mathcal{C} if and only if M and N are in \mathcal{C} .
- (3) If $0 \rightarrow M'' \rightarrow M \rightarrow M' \rightarrow 0$ is exact and $M' \in \mathcal{C}$, then $M \in \mathcal{C}$ if and only if $M'' \in \mathcal{C}$.

Thus, \mathcal{C} is closed under extensions and syzygies, and contains $\mathcal{P}(R)$.

Important example 2.3. *The following categories are resolving :*

- (1) $\mathcal{P}(R)$
- (2) $\text{mod}(R)$
- (3) Gorenstein dimension 0 modules (also known as totally reflexive modules).
- (4) $\{N \mid \text{Ext}^{>n}(N, M) = 0 \forall M \in \mathcal{X}\}$ where $\mathcal{X} \subset \text{mod}(R)$ and $n \geq 0$
- (5) $\{N \mid \text{Tor}^{>n}(N, M) = 0 \forall M \in \mathcal{X}\}$ where $\mathcal{X} \subset \text{mod}(R)$ and $n \geq 0$.
- (6) $MCM(R)$ when R is Cohen-Macaulay and local

A special class of resolving subcategories are thick subcategories.

Definition 2.4. Let $\mathcal{X} \subset \text{mod}(R)$. A full subcategory $\mathcal{C} \subset \mathcal{X}$ is a thick subcategory of \mathcal{X} (or \mathcal{C} is thick in \mathcal{X}) if it is resolving and for any exact sequence with $M, M', M'' \in \mathcal{X}$,

$$0 \rightarrow M'' \rightarrow M \rightarrow M' \rightarrow 0$$

$M'', M \in \mathcal{C}$ implies $M' \in \mathcal{C}$ too.

Some authors do not require that thick subcategories contain R . However, our definition is standard amongst certain authors, for example [27]. The intuition behind requiring R to be in a thick subcategory traces back to the stable category of maximal Cohen-Macaulay modules over a Gorenstein ring. More on the nomenclature and concepts can be found in [27].

Lemma 2.5. *For a subset $\mathcal{C} \subset \text{mod}(R)$, we let $\bar{\mathcal{C}}$ denote the category of modules M such that $\dim_{\mathcal{C}} M$ is finite. If \mathcal{C} is resolving, then*

$$\bar{\mathcal{C}} = \{M \in \text{mod}(R) \mid \Omega^{\gg 0} M \in \mathcal{C}\}$$

and $\bar{\mathcal{C}}$ is thick in $\text{mod}(R)$ (where $\Omega^i M$ is the i -th syzygy of M).

The proof is standard and is left to the reader but can be found in [21, Corollary 2.6 and Corollary 2.9].

Important example 2.6. *The following categories are obtained as the thick closures in $\text{mod}(R)$ of the resolving categories in Example 2.3 .*

- (1) *The category of modules with finite projective dimension, which we will denote by $\mathcal{P}(R)$.*
- (2) *$\text{mod}(R)$*
- (3) *The category of modules with finite Gorenstein dimension.*
- (4) *For any $\mathcal{X} \subset \text{mod}(R)$, $\{Y \mid \text{Ext}^{\gg 0}(Y, X) = 0 \ \forall X \in \mathcal{X}\}$*
- (5) *For any $\mathcal{X} \subset \text{mod}(R)$, $\{Y \mid \text{Tor}_{\gg 0}(Y, X) = 0 \ \forall X \in \mathcal{X}\}$*
- (6) *When R is Cohen-Macaulay and local, $\overline{\text{MCM}(R)} = \text{mod}(R)$.*

We define Serre subcategories so that it is convenient to talk about derived categories with supports.

Definition 2.7. A Serre subcategory \mathcal{L} of an Abelian category is a full subcategory such that if $0 \rightarrow M'' \rightarrow M \rightarrow M' \rightarrow 0$ is a short exact sequence in the ambient category, then $M \in \mathcal{L}$ iff $M', M'' \in \mathcal{L}$.

Remark 2.8. In fact every Serre subcategory of $\text{mod}(R)$ is obtained as the subcategory of modules supported on a specialization-closed subset V in $\text{Spec}(R)$ (i.e. if $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{p} \in V$, then $\mathfrak{q} \in V$) [9] . For example, \mathcal{M}_{fl} is obtained with $V = \text{MaxSpec}(R)$. Because of this, if a module M is in a Serre subcategory \mathcal{L} , then $\frac{R}{\text{Ann}(M)}$ is also in \mathcal{L} .

Let V be a closed subset of $\text{Spec}(R)$ and c be any integer. The main example we will consider in this article is the full subcategory of $\text{mod}(R)$ with modules :

$$\mathcal{L} = \{M \mid \text{codim}(\text{Supp}(M)) \geq c, \text{Supp}(M) \subseteq V\}.$$

Definition 2.9. A Serre subcategory satisfies condition (*) if given an ideal I in R such that $\frac{R}{I} \in \mathcal{L}$, there exists a regular sequence $(a_1, a_2, \dots, a_c) \in I$ such that $\frac{R}{(a_1, a_2, \dots, a_c)} \in \mathcal{L}$.

Important example 2.10. *The following Serre subcategories \mathcal{L} of $\text{mod}(R)$ satisfy condition (*) :*

- (1) *If R is equicodimensional (i.e. every maximal ideal has the same height) and Cohen-Macaulay, $\mathcal{L} = \mathcal{M}_{fl}$.*
- (2) *If R is Cohen-Macaulay, V is a set theoretic complete intersection, i.e. there exists a complete intersection ideal $J = (b_1, b_2, \dots, b_r)$ such that $V = V(J)$, c is any integer, $\mathcal{L} = \{M \mid \text{codim}(\text{Supp}(M)) \geq c, \text{Supp}(M) \subseteq V\}$.*
- (3) *An important special case of the previous example is when $V = \text{Spec}(R) = V(\emptyset)$. Then \mathcal{L} is the category of modules supported in codimension at least c .*
- (4) *V is a set theoretic complete intersection, $\mathcal{L} = \{M \mid \text{Supp}(M) \subseteq V\}$.*

Remark 2.11. We emphasize an immediate consequence of the definitions 2.4 and 2.7. For a thick subcategory \mathcal{T} of $\text{mod}(R)$ and a Serre subcategory \mathcal{L} of $\text{mod}(R)$ the intersection $\mathcal{T} \cap \mathcal{L}$ has all the properties of a thick subcategory except possibly that it contains R . In particular it has the 2-out-of-3 property, that is, in a short exact sequence in $\text{mod}(R)$, if two modules are in $\mathcal{T} \cap \mathcal{L}$, then so is the third.

For the rest of the article, we fix a resolving subcategory $\mathcal{A} \subseteq \text{mod}(R)$, a thick subcategory \mathcal{T} of $\text{mod}(R)$ and a Serre subcategory \mathcal{L} of $\text{mod}(R)$ satisfying condition (*).

Definition 2.12. Let $\mathcal{E} \subset \text{mod}(R)$ be a subcategory closed under extensions (e.g. $\mathcal{A}, \mathcal{T}, \mathcal{T} \cap \mathcal{L}$). Let $\text{Ch}_{\mathcal{L}}^b(\mathcal{E})$ be the subcategory of $\text{Ch}^b(\mathcal{E})$ consisting of complexes whose homologies lie in \mathcal{L} , and let $D_{\mathcal{L}}^b(\mathcal{E})$ be the subcategory of $D^b(\mathcal{E})$ consisting of objects whose homologies lie in \mathcal{L} .

Since \mathcal{L} is a Serre subcategory, it is easy to see that $D_{\mathcal{L}}^b(\mathcal{E})$ is a triangulated subcategory of $D^b(\mathcal{E})$. Furthermore, it is easy to show that $D_{\mathcal{L}}^b(\mathcal{E})$ is equivalent to the category obtained by inverting the quasi-isomorphisms of $\text{Ch}_{\mathcal{L}}^b(\mathcal{E})$.

Remark 2.13. We briefly mention what these definitions will help us achieve :

There is a natural functor $\text{Ch}^b(\mathcal{T} \cap \mathcal{L}) \rightarrow \text{Ch}_{\mathcal{L}}^b(\mathcal{T})$ which induces the natural functor $D^b(\mathcal{T} \cap \mathcal{L}) \rightarrow D_{\mathcal{L}}^b(\mathcal{T})$. In Theorem 4.5, we will prove that this is an equivalence.

When $\mathcal{T} = \overline{\mathcal{A}}$ where \mathcal{A} is resolving, there is a natural functor $\text{Ch}_{\mathcal{L}}^b(\mathcal{A}) \rightarrow \text{Ch}_{\mathcal{L}}^b(\mathcal{T})$ which induces an equivalence $D_{\mathcal{L}}^b(\mathcal{A}) \xrightarrow{\sim} D_{\mathcal{L}}^b(\mathcal{T})$.

To a reader who maybe lost in the notation, we highlight the two important special cases which might shed further light on what this achieves. Let R be equicodimensional and Cohen-Macaulay.

- (1) $\mathcal{A} = \mathcal{P}(R)$, $\mathcal{T} = \overline{\mathcal{A}} = \text{finite projective dimension modules}$, $\mathcal{L} = \mathcal{M}_{fl}$. Then $\overline{\mathcal{P}(R)} \cap \mathcal{M}_{fl} = \mathcal{T} \cap \mathcal{L}$ and hence the equivalences in 2.13 yield $D_{fl}^b(\mathcal{P}(R)) \simeq D^b(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{fl})$.
- (2) $\mathcal{A} = \text{mod}(R)$, $\mathcal{T} = \overline{\mathcal{A}} = \text{mod}(R)$, $\mathcal{L} = \mathcal{M}_{fl}$. Then the comparison in 2.13 yields the well-known equivalence $D_{fl}^b(\text{mod}(R)) \simeq D^b(\mathcal{M}_{fl})$.

PRELIMINARIES : CHAIN COMPLEXES IN \mathcal{A}

In this section, we state and prove some general lemmas for chain complexes of resolving (and thick) subcategories and their derived categories. We begin by introducing some notations for chain complexes.

Definition 2.14. Let $P_{\bullet} \in \text{Ch}^b(\text{mod}(R))$, i.e. chain complexes with elements in $\text{mod}(R)$.

- (1) $\min_c(P_{\bullet})$ is defined as $\sup\{n \mid P_i = 0 \ \forall i < n\}$.
- (2) $\max_c(P_{\bullet})$ is defined as $\inf\{n \mid P_i = 0 \ \forall i > n\}$.
- (3) $\min(P_{\bullet}) = \sup\{n \mid H_i(P_{\bullet}) = 0 \ \forall i < n\}$
- (4) $\max(P_{\bullet}) = \inf\{n \mid H_i(P_{\bullet}) = 0 \ \forall i > n\}$
- (5) $\text{Width}(P_{\bullet}) = \max(P_{\bullet}) - \min(P_{\bullet})$ if P_{\bullet} is not acyclic and $\text{Width}(P_{\bullet}) = 0$ if it is acyclic.
- (6) $Z_n^P = \ker(\partial_n)$, $B_n^P = \partial_{n+1}(P_{n+1})$.
- (7) $\text{Supph}(P_{\bullet}) = \{n \mid H_n(P_{\bullet}) \neq 0\}$.

(8) T denotes the shift functor.

The following lemma allows us to assume that the complexes we will work with have certain properties, simplifying later proofs.

Lemma 2.15. (1) Let $P_\bullet \in \text{Ch}^b(\mathcal{T} \cap \mathcal{L})$ and $\min(P_\bullet) \geq m$. Then there exists a quasi-isomorphism $P'_\bullet \xrightarrow{\phi} P_\bullet$ with $P'_\bullet \in \text{Ch}^b(\mathcal{T} \cap \mathcal{L})$, $\min_c(P'_\bullet) = m$.
 (2) Let $P_\bullet \in \text{Ch}^b_{\mathcal{L}}(\mathcal{T})$. Let t be any natural number and m be an integer such that $\min(P_\bullet) \geq m$. Then there exists a quasi-isomorphism $U_\bullet \xrightarrow{\phi} P_\bullet$ with $U_\bullet \in \text{Ch}^b_{\mathcal{L}}(\mathcal{T})$ such that U_i is free for all $i < t$ and $\min_c(U_\bullet) = m$.
 (3) Let $P_\bullet \in \text{Ch}^b_{\mathcal{L}}(\mathcal{T})$. Let m be an integer such that $\min(P_\bullet) \geq m$. Then there exists a quasi-isomorphism $U_\bullet \xrightarrow{\phi} P_\bullet$ such that U_i is free for all i .
 (4) Let \mathcal{E} be either \mathcal{T} or $\mathcal{T} \cap \mathcal{L}$. For $P_\bullet \in \text{Ch}^b(\mathcal{E})$, if $\text{Supph}(P_\bullet) = \{m\}$, then P_\bullet is isomorphic in $D^b(\mathcal{E})$ to the complex $T^m H_m(P_\bullet)$ in $D^b(\mathcal{E})$.

Proof. First, we prove (1). Consider the complex

$$P'_\bullet : \cdots \rightarrow P_{m+2} \rightarrow P_{m+1} \rightarrow Z_m^P \rightarrow 0 \cdots$$

Since $\min(P_\bullet) \geq m$, the complex

$$0 \rightarrow Z_m^P \rightarrow P_m \rightarrow \cdots \rightarrow P_{\min_c(P_\bullet)} \rightarrow 0$$

is acyclic. Since each $P_i \in \mathcal{T} \cap \mathcal{L}$ and $\mathcal{T} \cap \mathcal{L}$ has the 2-out-of-3 property by Remark 2.11, we conclude that Z_m^P is also in $\mathcal{T} \cap \mathcal{L}$. Hence P'_\bullet is in $\text{Ch}^b(\mathcal{T} \cap \mathcal{L})$. Furthermore, the inclusion map $P'_\bullet \rightarrow P_\bullet$ is a quasi-isomorphism. By construction, $\min_c(P'_\bullet) = m$ which completes the proof.

Now we prove (2). The statement is clearly true when P_\bullet is acyclic. Assume now that P_\bullet is not acyclic. The same arguments as in the previous paragraph show that there exists a quasi-isomorphism $P'_\bullet \xrightarrow{\phi'} P_\bullet$ with $P'_\bullet \in \text{Ch}^b_{\mathcal{L}}(\mathcal{T})$, $\min_c(P'_\bullet) = m$. Set $n = t - m$. For each i , choose free modules $Q_{i,j}$ for $0 \leq j \leq n$ such that

$$0 \rightarrow \Omega^n P'_i \rightarrow Q_{i,n} \rightarrow \cdots \rightarrow Q_{i,0} \rightarrow P'_i \rightarrow 0$$

is exact. Setting $Q_{i,n+1} = \Omega^n P'_i$ and $Q_{i,l} = 0$ for all $j \neq 0, \dots, n+1$ gives us a complex $Q_{i,\bullet}$ which is a resolution of P'_i . The differentials of P'_\bullet lift to give a double complex $Q_{\bullet,\bullet}$. Let U_\bullet be the total complex of $Q_{\bullet,\bullet}$. There is a quasi-isomorphism $U_\bullet \xrightarrow{\alpha} P'_\bullet$. Since \mathcal{T} is thick, each $\Omega^n P'_i$ is in \mathcal{T} , and so $U_\bullet \in \text{Ch}^b_{\mathcal{L}}(\mathcal{T})$. The composition $U_\bullet \xrightarrow{\alpha} P'_\bullet \xrightarrow{\phi'} P_\bullet$ is the desired quasi-isomorphism.

Statement (3) is standard and follows from the proof of (2) by taking $t = \infty$.

Lastly, we show (4). By the proof of statements (1) and (2), there exists a quasi isomorphism $P'_\bullet \rightarrow P_\bullet$ such that $\min_c(P'_\bullet) = m$. Since both \mathcal{T} and \mathcal{L} have the 2-out-of-3 property (by Remark 2.11), $H_m(P'_\bullet)$ is in \mathcal{E} . Since $Z_m^{P'} = P'_m$, we have a map $P'_m \rightarrow H_m(P'_\bullet) \cong H_m(P_\bullet)$. This extends to a quasi-isomorphism $P'_\bullet \rightarrow T^m H_m(P_\bullet)$. This completes the proof. \square

We now state a result on morphisms in $D^b_{\mathcal{L}}(\mathcal{T})$.

Lemma 2.16. Let $P_\bullet, Q_\bullet \in D^b_{\mathcal{L}}(\mathcal{T})$ such that $\min(P_\bullet) > \max(Q_\bullet)$. Then we have $\text{Hom}_{D^b_{\mathcal{L}}(\mathcal{T})}(P_\bullet, Q_\bullet) = 0$.

Proof. The result is clear when P_\bullet is acyclic, and hence we assume that P_\bullet is not acyclic. Let $f \in \text{Hom}_{D_{\mathcal{Z}}^b(\mathcal{T})}(P_\bullet, Q_\bullet)$ be represented by a roof diagram

$$P_\bullet \xleftarrow{q} P'_\bullet \xrightarrow{g} Q_\bullet$$

where q is a quasi-isomorphism. Set $t = \max_c(Q_\bullet)$ and $m = \min_c(P_\bullet)$. By Lemma 2.15(2), we can assume that $\min_c(P'_\bullet) = m$ and P'_i is free for all $t \geq i \geq m$. Set $h_i = 0 : P'_i \rightarrow Q_{i+1}$ for all i not in $[t, m]$. Let ∂^Q be the differential of Q_\bullet . Since for all $t \geq i \geq m$, P'_i is free and $m > \max_c(Q_\bullet)$,

$$Q_{i+1} \xrightarrow{\partial_{i+1}^Q} \ker(\partial_i^Q)$$

is surjective, we can define maps $h_i : P'_i \rightarrow Q_{i+1}$ such that $h = \{h_i\}$ is a null-homotopy for g . Hence g is null-homotopic and $f = 0$. \square

We end this section with a general result on triangulated categories.

Lemma 2.17. *Let \mathcal{U}, \mathcal{V} be triangulated categories and $\xi : \mathcal{U} \rightarrow \mathcal{V}$ a triangulated functor. If ξ is full, essentially surjective, and faithful on objects, then ξ is an equivalence of categories.*

Proof. It suffices to show that ξ is faithful. Suppose $f : X \rightarrow Y$ is a morphism in \mathcal{U} such that $\xi(f) = 0$. Letting T be the shift functor, complete f to a triangle

$$C \xrightarrow{g} X \xrightarrow{f} Y \xrightarrow{h} TC.$$

In \mathcal{V} , this maps to the triangle

$$\xi(C) \xrightarrow{\xi(g)} \xi(X) \xrightarrow{0} \xi(Y) \xrightarrow{\xi(h)} T\xi(C).$$

So, by [17, Corollary 1.2.7], there exists an $\eta : \xi(X) \rightarrow \xi(C)$ such that $\xi(g) \circ \eta$ is the identity on $\xi(X)$. Since ξ is full, there exists a morphism $\tilde{\eta} : X \rightarrow C$ such that $\xi(\tilde{\eta}) = \eta$. Extending the morphism $g \circ \tilde{\eta}$ to an exact triangle in \mathcal{U} gives us the triangle

$$X \xrightarrow{g \circ \tilde{\eta}} X \rightarrow W \rightarrow TX.$$

Since $\xi(g \circ \tilde{\eta})$ is the identity on $\xi(X)$, the image of the above triangle in \mathcal{V} is

$$\xi(X) \xrightarrow{\text{Id}_{\xi(X)}} \xi(X) \rightarrow \xi(W) \rightarrow T\xi(X).$$

This means that $\xi(W) \cong 0$ in \mathcal{V} . But since ξ is faithful on objects, $W \cong 0$ in \mathcal{U} . Hence $g \circ \tilde{\eta}$ is an isomorphism in \mathcal{U} . So $\tilde{\eta} \circ (g \circ \tilde{\eta})^{-1} : X \rightarrow C$ gives a splitting, i.e. $g \circ \tilde{\eta} \circ (g \circ \tilde{\eta})^{-1}$ is the identity on X . Therefore, by [17, Corollary 1.2.8], the triangles

$$C \xrightarrow{g} X \xrightarrow{f} Y \xrightarrow{h} TC$$

$$T^{-1}Y \oplus X \rightarrow X \xrightarrow{0} Y \rightarrow Y \oplus TX$$

are isomorphic in \mathcal{U} . Hence $f = 0$ which completes the proof. \square

PRELIMINARIES : SEMIDUALIZING MODULES

Semidualizing modules seem to have arisen in work of Foxby and we refer to the exposition [22] by Sather-Wagstaff for more details. We will require them only when we consider dualities in the context of Witt groups in Section 5 and the reader can skip this subsection presently and refer back later.

Definition 2.18. A module C is semidualizing if $\text{Ext}^{>0}(C, C) = 0$ and the natural map $R \xrightarrow{\sim} \text{Hom}(C, C)$ is an isomorphism.

We fix a semidualizing module $C \in \text{mod}(R)$ and write $M^\dagger = \text{Hom}(M, C)$ for the rest of this subsection.

Definition 2.19. A finitely generated module M is totally C -reflexive if it satisfies the following

- (1) $\text{Ext}^{>0}(M, C) = 0$
- (2) $\text{Ext}^{>0}(M^\dagger, C) = 0$
- (3) The natural evaluation map $\eta_M : M \rightarrow M^{\dagger\dagger}$ is an isomorphism.

Let \mathcal{G}_C denote the category of totally C -reflexive modules. Then \mathcal{G}_C is a resolving subcategory.

Important example 2.20. Some examples of semidualizing modules C and the corresponding categories \mathcal{G}_C are :

- (1) When $C = R$, \mathcal{G}_R is simply the category of Gorenstein dimension 0 modules (also known as totally reflexive modules).
- (2) When R is local and $C = D$, that is, the dualizing (or canonical) module, then $\mathcal{G}_D = \text{MCM}(R)$.

Lemma 2.21. If $M \in \overline{\mathcal{G}_C}$, then $\dim_{\mathcal{G}_C} M = \min\{n \mid \text{Ext}^{>n}(M, C) = 0\}$.

3. THE KOSZUL CONSTRUCTION

We begin this section with a theorem based on a very interesting construction on Koszul complexes which essentially first appears in an unpublished preprint [7]. The construction is used in [19, Lemma 1] and is generalized in [8]. **Recall that $\mathcal{T} \subseteq \text{mod}(R)$ is a thick subcategory, and $\mathcal{L} \subseteq \text{mod}(R)$ is a Serre subcategory satisfying condition (*)**.

Theorem 3.1. Let $P_\bullet \in \text{Ch}_{\mathcal{L}}^b(\text{mod}(R))$ and with $\min(P_\bullet) \geq m$. Fix an ideal $J \subseteq R$ such that $\frac{R}{J} \in \mathcal{L}$. There exists a regular sequence $g_1, \dots, g_c \in J$ such that, after setting

$$K_\bullet = \text{Kos}(f_1, f_2, \dots, f_c) \otimes_R F,$$

for some free module F , there exists a chain map $\alpha : K_\bullet \rightarrow P_\bullet$ such that the following hold:

- $K_\bullet \in \text{Ch}_{\mathcal{L}}^b(\mathcal{P}(R))$
- $\min_c(K_\bullet) = m$
- $\text{Supph}(K_\bullet) = \{m\}$
- $H_m(\alpha) : H_m(K_\bullet) \rightarrow H_m(P_\bullet)$ is surjective.

The proof of our main result involves several technical subtleties, thus we require α and K_\bullet to have these specific properties. Since neither [7, 19, 8] give the exact statement needed, we give the following proof.

Proof. By Remark 2.8, we know that there exists a specialization closed subset $V \subseteq \text{Spec}(R)$ such that

$$\mathcal{L} = \{M \in \text{mod}(R) \mid \text{Supp}(M) \subseteq V\}.$$

Without loss of generality, we may assume that $m = 0$.

By Lemma 2.15(3), there exists a quasi-isomorphism $\pi : F_\bullet \rightarrow P_\bullet$ with each F_i free and $F_i = 0$ for all $i < 0$. By repeated application of the dual of [32, Corollary 10.4.7], we get the following natural isomorphisms of Hom spaces :

$$\text{End}_{K(R)}(F_\bullet) \cong \text{End}_{D(R)}(F_\bullet) \cong \text{Hom}_{D(R)}(F_\bullet, P_\bullet) \cong \text{Hom}_{K(R)}(F_\bullet, P_\bullet).$$

Since P_\bullet is a bounded complex, $\text{Hom}_{K(R)}(F_\bullet, P_\bullet)$ is a subquotient of a finitely generated module and hence is finitely generated. Hence, $\text{End}_{K(R)}(F_\bullet)$ is a finitely generated R -module. Similarly using the above isomorphisms and the boundedness of P_\bullet , we see that the map $(\text{End}_{K(R)}(F_\bullet))_p \rightarrow \text{End}_{K(R_p)}((F_\bullet)_p)$ is injective for every $p \in \text{Spec}(R)$.

Assume that $(\text{End}_{K(R)}(F_\bullet))_p \neq 0$ for some $p \in \text{Spec}(R)$. Then $\text{End}_{K(R_p)}((F_\bullet)_p) \neq 0$ which implies that $(F_\bullet)_p$ is not homotopic to 0_\bullet . Since F_\bullet is a bounded below complex of free modules, $(F_\bullet)_p$ is not acyclic, and so $H_i(F_\bullet)_p \neq 0$ for some $i \in \mathbb{N}$. Therefore we have

$$\text{Supp}(\text{End}_{K(R)}(F_\bullet)) \subseteq \bigcup_{i=0}^{\infty} \text{Supp}(H_i(F_\bullet)) = \bigcup_{i=0}^{\infty} \text{Supp}(H_i(P_\bullet)) \subseteq V$$

where the last containment follows from the assumption that $P_\bullet \in \text{Ch}_{\mathcal{L}}^b(\text{mod}(R))$. Set $I = \text{Ann}(\text{End}_{K(R)}(F_\bullet))$. We have

$$\text{Supp}\left(\frac{R}{I \cap J}\right) = \text{Supp}\left(\frac{R}{I}\right) \cup \text{Supp}\left(\frac{R}{J}\right) = \text{Supp}(\text{End}_{K(R)}(F_\bullet)) \cup \text{Supp}\left(\frac{R}{J}\right) \subseteq V.$$

It follows that $\frac{R}{I \cap J} \in \mathcal{L}$.

Invoking condition (*), there exists a regular sequence $f_1, \dots, f_c \in I \cap J$ such that $\frac{R}{(f_1, f_2, \dots, f_c)} \in \mathcal{L}$. We show that this is our desired regular sequence. Set

$$K_\bullet = \text{Kos}(f_1, f_2, \dots, f_c) \otimes_R F_0.$$

For each $j \in \{1, \dots, c\}$, since $f_j \in I$, $f_j \text{Id}_{F_\bullet}$ is null-homotopic with null-homotopy α_j . Letting $-^*$ denote the R -dual of a complex, it follows that $f_j \text{Id}_{F_\bullet^*}$ is also null-homotopic with null-homotopy α_j^* .

The result [8, Proposition 23] states that when F_\bullet is bounded above, there exists a chain map $\phi : F_\bullet^* \rightarrow K_\bullet^*$ such that ϕ_0 is an isomorphism (in the language of [8], $(\alpha_1^*, \dots, \alpha_c^*)$ is an S -contraction with weight (f_1, f_2, \dots, f_c)). However, a close examination of the proof reveals that the bounded assumption is essential (the authors only use the bounded assumption earlier in the paper to construct an S -contraction). Thus, even though F_\bullet is not bounded above, such a ϕ exists.

Set $\tilde{\alpha} = \phi^* : K_\bullet \rightarrow F_\bullet$. Since $\tilde{\alpha}_0$ is an isomorphism, and since $K_{-1} = F_{-1} = 0$, the commutative diagram

$$\begin{array}{ccc} K_0 & \xrightarrow[\tilde{\alpha}_0]{\sim} & F_0 \\ \downarrow & & \downarrow \\ H_0(K_\bullet) & \xrightarrow{H_0(\tilde{\alpha})} & H_0(F_\bullet) \end{array}$$

shows that $H_0(\tilde{\alpha}) : H_0(K_\bullet) \rightarrow H_0(F_\bullet)$ is surjective. Set $\alpha = \pi \circ \tilde{\alpha} : K_\bullet \rightarrow P_\bullet$. Since π is a quasi-isomorphism, it is clear that α and K_\bullet have the desired properties. \square

The following result will also be useful in applying Theorem 3.1 in the proof of Proposition 4.4.

Lemma 3.2. *Assume the set up of Theorem 3.1 and consider the chain map $\alpha : K_\bullet \rightarrow P_\bullet$ constructed there. Extend this chain map to the exact triangle*

$$T^{-1}C_\bullet \rightarrow K_\bullet \xrightarrow{\alpha} P_\bullet \rightarrow C_\bullet.$$

If $\text{Width}(P_\bullet) > 0$, then we have the following:

- (1) $\text{Width}(C_\bullet) < \text{Width}(P_\bullet)$
- (2) $\text{Width}(T^{-1}C_\bullet \oplus K_\bullet) < \text{Width}(P_\bullet)$.

Proof. Set $m = \min(P_\bullet)$. Since $H_i(K_\bullet) = 0$ for all $i \neq m$, we have $H_i(P_\bullet) \cong H_i(C_\bullet)$ for all $i \neq m, m+1$. Furthermore, $H_m(\alpha)$ is surjective so $H_m(C_\bullet) = 0$. The above statements now follow from these observations. \square

In other works such as [7],[8], and [19], the authors use results similar to Theorem 3.1 to give isomorphisms of Grothendieck groups. Since we intend to prove an equivalence of categories, we need a version of Theorem 3.1 for morphisms of complexes.

Theorem 3.3. *Let $X_\bullet \xrightarrow{g} Y_\bullet$ be a morphism in $D_{\mathcal{L}}^b(\mathcal{T})$ such that X_\bullet, Y_\bullet are complexes in $\text{Ch}^b(\mathcal{T} \cap \mathcal{L})$, $\min(X_\bullet \oplus Y_\bullet) = m$, and $\min_c(X_\bullet), \min_c(Y_\bullet) \geq m$. Then there exist complexes M_\bullet^X and M_\bullet^Y in $\text{Ch}^b(\mathcal{T} \cap \mathcal{L})$ and chain maps $M_\bullet^X \xrightarrow{\beta^X} X_\bullet$, $M_\bullet^Y \xrightarrow{\beta^Y} Y_\bullet$ and $M_\bullet^X \xrightarrow{g'} M_\bullet^Y$ such that*

- $M_\bullet^X, M_\bullet^Y \in \text{Ch}^b(\mathcal{T} \cap \mathcal{L})$ with $M_i^X = M_i^Y = 0$ for all $i \neq m$.
- there is a commutative square in $D_{\mathcal{L}}^b(\mathcal{T})$:

$$\begin{array}{ccc} M_\bullet^X & \xrightarrow{\beta^X} & X_\bullet \\ \kappa \downarrow & & \downarrow g \\ M_\bullet^Y & \xrightarrow{\beta^Y} & Y_\bullet \end{array}$$

- $H_m(\beta^X)$ and $H_m(\beta^Y)$ are surjective.

Proof. Let M_\bullet^Y be the complex $T^m Y_m$. Then $M_\bullet^Y \in \text{Ch}^b(\mathcal{T} \cap \mathcal{L})$ and clearly there is a chain complex morphism $\beta^Y : M_\bullet^Y \rightarrow Y_\bullet$. Since $H_m(M_\bullet^Y) = Y_m$ and $H_m(Y_\bullet)$ is a quotient of Y_m , we obtain that $H_m(\beta^Y)$ is surjective.

Let g be given by a roof diagram $X_\bullet \xleftarrow{q} Q_\bullet \xrightarrow{f} Y_\bullet$ where q is a quasi-isomorphism. By Lemma 2.15(2), we may assume $\min_c(Q_\bullet) = m$. Let Q'_\bullet be the pull-back of α^Y and f in $\text{Ch}^b(\text{mod}(R))$:

$$\begin{array}{ccc} Q'_\bullet & \xrightarrow{\nu} & Q_\bullet \\ \mu \downarrow & & \downarrow f \\ M_\bullet^Y & \xrightarrow{\beta^Y} & Y_\bullet \end{array}$$

By definition of the pull-back, there is a left exact sequence of complexes :

$$0_\bullet \rightarrow Q'_\bullet \rightarrow Q_\bullet \oplus M_\bullet^Y \xrightarrow{(f, -\beta^Y)} Y_\bullet.$$

Let B_\bullet be the image of $(f, -\alpha^Y)$. Since \mathcal{L} is closed under subobjects and each $B_i \subset Y_i \in \mathcal{L}$, we have $B_\bullet \in \text{Ch}^b(\mathcal{L})$. Furthermore, since \mathcal{L} is also closed under quotients, $B_\bullet \in \text{Ch}^b_{\mathcal{L}}(\text{mod}(R))$. Since the homologies of $Q_\bullet \oplus M_\bullet^Y$ also lie in \mathcal{L} , the short exact sequence

$$0_\bullet \rightarrow Q'_\bullet \rightarrow Q_\bullet \oplus M_\bullet^Y \rightarrow B_\bullet \rightarrow 0_\bullet$$

shows that $Q'_\bullet \in \text{Ch}^b_{\mathcal{L}}(\text{mod}(R))$. Set $\lambda = q \circ \nu$.

Set $J = \text{Ann}(X_m)$. By Remark 2.8, $\frac{R}{J} \in \mathcal{L}$. By Theorem 3.1, there exists a regular sequence $x_1, x_2, \dots, x_c \in J$ and a chain map $\alpha^X : K_\bullet^X \rightarrow Q'_\bullet$ with the following properties :

- $K_\bullet^X = \text{Kos}(x_1, x_2, \dots, x_c) \otimes_R R^n \in \text{Ch}^b_{\mathcal{L}}(\mathcal{P})$
- $\min_c(K_\bullet^X) = m$
- $\text{Supp}(K_\bullet^X) = \{m\}$
- $H_m(\alpha^X) : H_m(K_\bullet^X) \rightarrow H_m(Q'_\bullet)$ is surjective .

Thus, we obtain the following commutative diagram in $D^b_{\mathcal{L}}(\text{mod}(R))$

$$\begin{array}{ccccc} K_\bullet^X & \xrightarrow{\alpha^X} & Q'_\bullet & \xrightarrow{\lambda} & X_\bullet \\ & & \mu \downarrow & & \downarrow g \\ & & M_\bullet^Y & \xrightarrow{\beta^Y} & Y_\bullet \end{array}$$

Let M_\bullet^X be the chain complex $T^m H_m(K_\bullet^X) \in \text{Ch}^b(\bar{\mathcal{P}} \cap \mathcal{L}) \subseteq \text{Ch}^b(\mathcal{T} \cap \mathcal{L}) \subseteq \text{Ch}^b_{\mathcal{L}}(\mathcal{T})$. Since

$$\frac{K_m^X}{(x_1, \dots, x_c)K_m^X} \cong H_m(K_\bullet^X)$$

and $x_1, \dots, x_c \in J = \text{Ann}(X_m)$, the map $\lambda_m \circ \alpha_m^X : K_m^X \rightarrow X_m$ factors through $H_m(K_\bullet^X)$ giving us the homomorphism $\beta_m^X : H_m(K_\bullet^X) \rightarrow X_m$. Also, since $Z_m^{K^X} = K_m^X$, the image of $\lambda_m \circ \alpha_m^X$, and hence the image of β_m^X , lies in Z_m^X . Therefore, we have actually defined a chain map $\beta^X : M_\bullet^X \rightarrow X_\bullet$. Similarly, it is also clear that $\mu \circ \alpha^X$ factors through M_\bullet^X giving us the chain map $\kappa : M_\bullet^X \rightarrow M_\bullet^Y$. Hence we

have the required commutative diagram in $D_{\mathcal{L}}^b(\mathcal{T})$:

$$\begin{array}{ccc} M_{\bullet}^X & \xrightarrow{\beta^X} & X_{\bullet} \\ \kappa \downarrow & & \downarrow g \\ M_{\bullet}^Y & \xrightarrow{\beta^Y} & Y_{\bullet} \end{array}$$

It remains to show that $H_m(\beta^X)$ is surjective. Since q is a quasi-isomorphism and $H_m(\alpha^X)$ is surjective, in order to show that $H_m(\beta^X) = H_m(q) \circ H_m(\nu) \circ H_m(\alpha^X)$ is surjective, it suffices to show that $H_m(\nu)$ is surjective.

Since β_m^Y is an isomorphism and $\beta_{m+1}^Y = 0$, $Q'_m \xrightarrow{\nu_m} Q_m$ is an isomorphism and $Q'_{m+1} \xrightarrow{\nu_{m+1}} Q_{m+1}$ is injective. Since $Q'_i = Q_i = 0$ for $i < m$, we have the following morphism of exact sequences:

$$\begin{array}{ccccccc} Q'_{m+1} & \longrightarrow & Q'_m & \longrightarrow & H_m(Q'_{\bullet}) & \longrightarrow & 0 \\ \nu_{m+1} \downarrow & & \nu_m \downarrow & & H_m(\nu) \downarrow & & \\ Q_{m+1} & \longrightarrow & Q_m & \longrightarrow & H_m(Q_{\bullet}) & \longrightarrow & 0 \end{array}$$

Thus, $H_m(\nu)$ is surjective. This completes the proof. \square

The following result will also be useful in applying Theorem 3.1 in the proof of Proposition 4.4.

Lemma 3.4. *Assume the hypothesis of Theorem 3.3 and consider the commutative diagram constructed*

$$\begin{array}{ccc} M_{\bullet}^X & \xrightarrow{\beta^X} & X_{\bullet} \\ \downarrow & & \downarrow g \\ M_{\bullet}^Y & \xrightarrow{\beta^Y} & Y_{\bullet} \end{array}$$

Suppose $\text{Width}(X_{\bullet} \oplus Y_{\bullet}) = k > 0$. Complete β^X and β^Y to exact triangles :

$$T^{-1}C_{\bullet}^X \rightarrow M_{\bullet}^X \xrightarrow{\beta^X} X_{\bullet} \xrightarrow{\gamma^X} C_{\bullet}^X \quad T^{-1}C_{\bullet}^Y \rightarrow M_{\bullet}^Y \xrightarrow{\beta^Y} Y_{\bullet} \xrightarrow{\gamma^Y} C_{\bullet}^Y$$

Then we have the following :

- (1) $\text{Width}(C_{\bullet}^X \oplus C_{\bullet}^Y) < k$
- (2) $\text{Width}(C_{\bullet}^X \oplus Y_{\bullet}) \leq k$
- (3) If $\min(X_{\bullet}) < \min(Y_{\bullet})$, then $\min(C_{\bullet}^X \oplus Y_{\bullet}) \leq k - 1$.

Proof. The proof is straightforward and the same arguments in the proof of Lemma 3.2 imply the above statements. \square

4. THE EQUIVALENCE OF DERIVED CATEGORIES

In this section, we prove the promised equivalence of the derived categories $D^b(\mathcal{T} \cap \mathcal{L})$ and $D_{\mathcal{L}}^b(\mathcal{T})$ and then consider the case where $\mathcal{T} = \overline{\mathcal{A}}$, yielding our main theorem. We begin by defining the natural functor.

Definition 4.1. Let $\iota : D^b(\mathcal{T} \cap \mathcal{L}) \rightarrow D^b_{\mathcal{L}}(\mathcal{T})$ be the natural functor induced by the inclusion $\text{Ch}^b(\mathcal{T} \cap \mathcal{L}) \hookrightarrow \text{Ch}^b_{\mathcal{L}}(\mathcal{T})$. Abusing notation, we will sometimes write $X_{\bullet} \in D^b_{\mathcal{L}}(\mathcal{T})$ when we mean $\iota(X_{\bullet})$. Similarly, for a morphism $f \in D^b_{\mathcal{L}}(\mathcal{T})$, we will write $f \in D^b(\mathcal{T} \cap \mathcal{L})$ when we mean $f = \iota(g)$ for some $g \in D^b(\mathcal{T} \cap \mathcal{L})$.

Lemma 4.2. *Viewing the modules $M, N \in \mathcal{T} \cap \mathcal{L}$ as complexes, we have*

$$\text{Hom}_{\mathcal{T} \cap \mathcal{L}}(M, N) \xrightarrow{\sim} \text{Hom}_{D^b(\mathcal{T} \cap \mathcal{L})}(M, N) \xrightarrow{\sim} \text{Hom}_{D^b_{\mathcal{L}}(\mathcal{T})}(\iota(M), \iota(N)) \xrightarrow{\sim} \text{Hom}_R(M, N).$$

Proof. The maps are induced by the functors

$$\mathcal{T} \cap \mathcal{L} \rightarrow D^b(\mathcal{T} \cap \mathcal{L}) \xrightarrow{\iota} D^b_{\mathcal{L}}(\mathcal{T}) \xrightarrow{H_0} \text{mod}(R)$$

where the last functor is the zeroth homology functor. The composition of the maps is identity on $\text{Hom}_R(M, N)$. Thus the first map is injective. Let $\beta \in \text{Hom}_{D^b(\mathcal{T} \cap \mathcal{L})}(M, N)$. Let β be given by the roof diagram $M \xleftarrow{q} T_{\bullet} \xrightarrow{f} N$ where q is a quasi-isomorphism. By Lemma 2.15(1), there is a quasi-isomorphism $T'_{\bullet} \xrightarrow{i} T_{\bullet}$ such that $\min_c(T'_{\bullet}) = 0$. Then $H_0(T_{\bullet}) = H_0(T'_{\bullet})$ and let $T'_{\bullet} \xrightarrow{\mu} H_0(T_{\bullet})$ be the obvious map. Then the following diagram commutes :

$$\begin{array}{ccccc} & & T_{\bullet} & & \\ & \nearrow \sim & \uparrow i & \searrow f & \\ M & & T'_{\bullet} & & N \\ & \nwarrow \sim & \downarrow \mu & \nearrow f \circ i & \\ & & H_0(T_{\bullet}) & & \\ & \nearrow H_0(q) \circ \mu & & \searrow H_0(f) \circ H_0(q)^{-1} & \\ & & M & & \end{array}$$

This tells us that β is equivalent to the map induced by $H_0(f) \circ H_0(q)^{-1}$. So the first map is an isomorphism which forces the second map to be an injection. The same argument shows that the map

$$\text{Hom}_{\mathcal{T} \cap \mathcal{L}}(M, N) \rightarrow \text{Hom}_{D^b_{\mathcal{L}}(\mathcal{T})}(\iota(M), \iota(N))$$

is surjective, proving the claim. \square

Lemma 4.3. *Let $X_{\bullet}, Y_{\bullet} \in D^b(\mathcal{T} \cap \mathcal{L})$ such that $\text{Supp}(Y_{\bullet}) = \{m\}$ and $\min(X_{\bullet}) \geq m$. Then $\text{Hom}_{D^b(\mathcal{T} \cap \mathcal{L})}(X_{\bullet}, Y_{\bullet}) \rightarrow \text{Hom}_{D^b_{\mathcal{L}}(\mathcal{T})}(X_{\bullet}, Y_{\bullet})$ is injective.*

Proof. We may assume that $m = 0$. By Lemma 2.15(4), Y_{\bullet} is isomorphic in $D^b(\mathcal{T} \cap \mathcal{L})$ to the complex $U_{\bullet} = H_0(Y_{\bullet}) \in \mathcal{T} \cap \mathcal{L}$. Thus we choose to work with U_{\bullet} . Let $g \in \text{Hom}_{D^b(\mathcal{T} \cap \mathcal{L})}(X_{\bullet}, U_{\bullet})$. Then g is given by a roof diagram $X_{\bullet} \xleftarrow{q} W_{\bullet} \xrightarrow{f} U_{\bullet}$ where q is a quasi-isomorphism. It is enough to show that $\iota(f) = 0$ implies $f = 0$. By Lemma 2.15(1), we may assume that $\min_c(W_{\bullet}) = \min(X_{\bullet}) \geq 0$. Note that $\iota(f) = 0$ implies $H_0(f) = 0$. Since $\min_c(W_{\bullet}) \geq 0$, there is a natural surjection $W_0 \xrightarrow{h} H_0(W_{\bullet})$. Further, since $U_i = 0$ for all $i \neq 0$, $f_0 = H_0(f) \circ h = 0$. This finishes the proof. \square

We now have all the ingredients to prove the equivalence of the derived categories $D^b(\mathcal{T} \cap \mathcal{L})$ and $D^b_{\mathcal{L}}(\mathcal{T})$.

Proposition 4.4. *The functor ι is essentially surjective and full.*

Proof. We prove the following set of statements by induction on k :

- (1) For every $P_\bullet \in D_{\mathcal{L}}^b(\mathcal{T})$, with $\text{Width}(P_\bullet) = k$, there exists $\tilde{P}_\bullet \in D^b(\mathcal{T} \cap \mathcal{L})$ such that $\iota(\tilde{P}_\bullet) \cong P_\bullet$.
- (2) For every $X_\bullet, Y_\bullet \in D^b(\mathcal{T} \cap \mathcal{L})$ with $\text{Width}(X_\bullet \oplus Y_\bullet) = k$, the map $\text{Hom}_{D^b(\mathcal{T} \cap \mathcal{L})}(X_\bullet, Y_\bullet) \rightarrow \text{Hom}_{D_{\mathcal{L}}^b(\mathcal{T})}(X_\bullet, Y_\bullet)$ induced by ι is surjective.

Let $k = 0$. We have $\text{Supp}(P_\bullet) \subseteq \{m\}$ for some m . By Lemma 2.15(4), P_\bullet is isomorphic in $D_{\mathcal{L}}^b(\mathcal{T})$ to a shift of the module $H_m(P_\bullet)$. Since \mathcal{T} has the 2-out-of-3 property (by Remark 2.11), $H_m(P_\bullet)$ is in \mathcal{T} . Therefore, since $H_m(P_\bullet) \in \mathcal{L}$, we have $H_m(P_\bullet) \in \mathcal{T} \cap \mathcal{L}$, proving statement (1) for $k = 0$. We now prove (2). Since $k = 0$, $\text{Supp}(X_\bullet), \text{Supp}(Y_\bullet) \subseteq \{m\}$. Hence, by Lemma 2.15(4), X_\bullet and Y_\bullet are isomorphic in $D^b(\mathcal{T} \cap \mathcal{L})$ to $T^m H_m(X_\bullet)$ and $T^m H_m(Y_\bullet)$ where T is the shift functor. Statement (2) now follows from Lemma 4.2.

Now suppose $k > 0$ and statements (1) and (2) are true for all $k' < k$. Let $\min(P_\bullet) = m$. By Theorem 3.1, there exists a complex K_\bullet in $\text{Ch}_{\mathcal{L}}(\mathcal{T})$ and a chain map $K_\bullet \xrightarrow{\beta} P_\bullet$ such that $\text{Supp}(K_\bullet) = \{m\}$, $\min_c(K_\bullet) = m$ and $H_m(\beta)$ is surjective. Note by Lemma 2.15(4) K_\bullet and $T^m H_m(K_\bullet)$ are isomorphic in $D_{\mathcal{L}}^b(\mathcal{T})$. Extend β to an exact triangle

$$T^{-1}C_\bullet \xrightarrow{\alpha} K_\bullet \xrightarrow{\beta} P_\bullet \xrightarrow{\gamma} C_\bullet.$$

By Lemma 3.2, we have $\text{Width}(T^{-1}C_\bullet \oplus T^m H_m(K_\bullet)) = \text{Width}(T^{-1}C_\bullet \oplus K_\bullet) < k$ and $\text{Width}(C_\bullet) < k$. Therefore, by induction, there exists $\tilde{C}_\bullet \in D^b(\mathcal{T} \cap \mathcal{L})$ such that $\iota(\tilde{C}_\bullet) \cong C_\bullet$, and the following map is a surjection

$$\text{Hom}_{D^b(\mathcal{T} \cap \mathcal{L})}(T^{-1}\tilde{C}_\bullet, T^m H_m(K_\bullet)) \twoheadrightarrow \text{Hom}_{D_{\mathcal{L}}^b(\mathcal{T})}(\iota(T^{-1}\tilde{C}_\bullet), \iota(T^m H_m(K_\bullet))) \cong \text{Hom}_{D_{\mathcal{L}}^b(\mathcal{T})}(T^{-1}\tilde{C}_\bullet, K_\bullet)$$

Because of this surjection, there exists $\tilde{\alpha} : T^{-1}\tilde{C}_\bullet \rightarrow T^m H_m(K_\bullet)$ in $D^b(\mathcal{T} \cap \mathcal{L})$ with cone $\tilde{P}_\bullet \in D^b(\mathcal{T} \cap \mathcal{L})$ and maps $\tilde{\beta}$ and $\tilde{\gamma}$ so that the following diagram commutes

$$(1) \quad \begin{array}{ccccccc} T^{-1}C_\bullet & \xrightarrow{\alpha} & K_\bullet & \xrightarrow{\beta} & P_\bullet & \xrightarrow{\gamma} & C_\bullet \\ \downarrow \wr & & \downarrow \wr & & & & \downarrow \wr \\ \iota(T^{-1}\tilde{C}_\bullet) & \xrightarrow{\iota(\tilde{\alpha})} & \iota(H_m(K_\bullet)) & \xrightarrow{\iota(\tilde{\beta})} & \iota(\tilde{P}_\bullet) & \xrightarrow{\iota(\tilde{\gamma})} & \iota(\tilde{C}_\bullet) \end{array}$$

It follows that there is an isomorphism $\iota(\tilde{P}_\bullet) \cong P_\bullet$, proving statement (1).

We now prove statement (2). Let $X_\bullet, Y_\bullet \in D^b(\mathcal{T} \cap \mathcal{L})$ and $f \in \text{Hom}_{D_{\mathcal{L}}^b(\mathcal{T})}(X_\bullet, Y_\bullet)$. Set $m = \min(X_\bullet \oplus Y_\bullet)$. By Lemma 2.15(1), we may assume that $\min_c(X_\bullet), \min_c(Y_\bullet) \geq m$. From Theorem 3.3, we have chain complex maps $M_\bullet^X \xrightarrow{\beta^X} X_\bullet$, $M_\bullet^Y \xrightarrow{\beta^Y} Y_\bullet$ and $M_\bullet^X \xrightarrow{\kappa} M_\bullet^Y$ such that

- $M_\bullet^X, M_\bullet^Y \in \text{Ch}^b(\mathcal{T} \cap \mathcal{L})$ are concentrated in degree m
- $H_m(\beta^X)$ and $H_m(\beta^Y)$ are surjective.
- there is a commutative diagram in $D_{\mathcal{L}}^b(\mathcal{T})$:

$$\begin{array}{ccc} M_\bullet^X & \xrightarrow{\beta^X} & X_\bullet \\ \downarrow \kappa & & \downarrow f \\ M_\bullet^Y & \xrightarrow{\beta^Y} & Y_\bullet \end{array}$$

Taking cones $C_{\bullet}^X, C_{\bullet}^Y$ of β^X, β^Y respectively in $D^b(\mathcal{T} \cap \mathcal{L})$, we get a morphism of triangles in $D_{\mathcal{L}}^b(\mathcal{T})$ (as mentioned earlier, we drop the ι) :

$$(2) \quad \begin{array}{ccccccc} T^{-1}C_{\bullet}^X & \xrightarrow{\alpha^X} & M_{\bullet}^X & \xrightarrow{\beta^X} & X_{\bullet} & \xrightarrow{\gamma^X} & C_{\bullet}^X \\ \downarrow T^{-1}\lambda & & \downarrow \kappa & & \downarrow f & & \downarrow \lambda \\ T^{-1}C_{\bullet}^Y & \xrightarrow{\alpha^Y} & M_{\bullet}^Y & \xrightarrow{\beta^Y} & Y_{\bullet} & \xrightarrow{\gamma^Y} & C_{\bullet}^Y \end{array}$$

We emphasize again that all the objects are in $D^b(\mathcal{T} \cap \mathcal{L})$ and the horizontal maps and κ are morphisms from $D^b(\mathcal{T} \cap \mathcal{L})$. By Lemma 3.4, $\text{Width}(C_{\bullet}^X \oplus C_{\bullet}^Y) < k$, and so, by the induction hypothesis, there exists $\tilde{\lambda} \in \text{Hom}_{D^b(\mathcal{T} \cap \mathcal{L})}(C_{\bullet}^X, C_{\bullet}^Y)$ such that $\iota(\tilde{\lambda}) \simeq \lambda$. We now have a diagram in $D^b(\mathcal{T} \cap \mathcal{L})$ whose rows are exact triangles:

$$(3) \quad \begin{array}{ccccccc} T^{-1}C_{\bullet}^X & \xrightarrow{\alpha^X} & M_{\bullet}^X & \xrightarrow{\beta^X} & X_{\bullet} & \xrightarrow{\gamma^X} & C_{\bullet}^X \\ \downarrow T^{-1}\tilde{\lambda} & & \downarrow \kappa & & & & \downarrow \tilde{\lambda} \\ T^{-1}C_{\bullet}^Y & \xrightarrow{\alpha^Y} & M_{\bullet}^Y & \xrightarrow{\beta^Y} & Y_{\bullet} & \xrightarrow{\gamma^Y} & C_{\bullet}^Y \end{array}$$

A priori, the left square in diagram (3) may not commute. Note however that

$$\iota(\alpha^Y \circ T^{-1}\tilde{\lambda} - \kappa \circ \alpha^X) = \iota(\alpha^Y \circ T^{-1}\tilde{\lambda}) - \iota(\kappa \circ \alpha^X) = \alpha^Y \circ T^{-1}\lambda - \kappa \circ \alpha^X = 0.$$

By Lemma 4.3, the map $\text{Hom}_{D^b(\mathcal{T} \cap \mathcal{L})}(T^{-1}C_{\bullet}^X, M_{\bullet}^Y) \rightarrow \text{Hom}_{D_{\mathcal{L}}^b(\mathcal{T})}(T^{-1}C_{\bullet}^X, M_{\bullet}^Y)$ induced by ι is injective and so $\alpha^Y \circ T^{-1}\tilde{\lambda} - \kappa \circ \alpha^X = 0$. Hence the left square in diagram (3) commutes. Thus there exists $X_{\bullet} \xrightarrow{g} Y_{\bullet}$ which gives a morphism of triangles in $D^b(\mathcal{T} \cap \mathcal{L})$:

$$\begin{array}{ccccccc} T^{-1}C_{\bullet}^X & \xrightarrow{\alpha^X} & M_{\bullet}^X & \xrightarrow{\beta^X} & X_{\bullet} & \xrightarrow{\gamma^X} & C_{\bullet}^X \\ \downarrow T^{-1}\tilde{\lambda} & & \downarrow \kappa & & \downarrow g & & \downarrow \tilde{\lambda} \\ T^{-1}C_{\bullet}^Y & \xrightarrow{\alpha^Y} & M_{\bullet}^Y & \xrightarrow{\beta^Y} & Y_{\bullet} & \xrightarrow{\gamma^Y} & C_{\bullet}^Y \end{array}$$

This means if we replace f with $\iota(g)$ in diagram (2), the diagram remains commutative. Unfortunately, it does not follow from the third axiom of triangulated categories that $f = \iota(g)$.

Set $\delta = f - \iota(g)$. We now have a morphism of triangles in $D_{\mathcal{L}}^b(\mathcal{T})$.

$$\begin{array}{ccccccc} T^{-1}C_{\bullet}^X & \xrightarrow{\alpha^X} & M_{\bullet}^X & \xrightarrow{\beta^X} & X_{\bullet} & \xrightarrow{\gamma^X} & C_{\bullet}^X \\ \downarrow 0 & & \downarrow 0 & & \downarrow \delta & & \downarrow 0 \\ T^{-1}C_{\bullet}^Y & \xrightarrow{\alpha^Y} & M_{\bullet}^Y & \xrightarrow{\beta^Y} & Y_{\bullet} & \xrightarrow{\gamma^Y} & C_{\bullet}^Y \end{array}$$

Hence, there exists $X_\bullet \xrightarrow{u} K_\bullet^Y$ such that $\delta = \beta^Y \circ u$. Similarly, there exists a $C_\bullet^X \xrightarrow{v} Y_\bullet$ such that $\delta = v \circ \gamma^X$. Hence, we have

$$\begin{array}{ccccccc} T^{-1}C_\bullet^X & \xrightarrow{\alpha^X} & M_\bullet^X & \xrightarrow{\beta^X} & X_\bullet & \xrightarrow{\gamma^X} & C_\bullet^X \\ \downarrow 0 & & \downarrow 0 & \swarrow u & \downarrow \delta & \swarrow v & \downarrow 0 \\ T^{-1}C_\bullet^Y & \xrightarrow{\alpha^Y} & M_\bullet^Y & \xrightarrow{\beta^Y} & Y_\bullet & \xrightarrow{\gamma^Y} & C_\bullet^Y \end{array}$$

We break the proof into three cases.

- Case (i) If $\min(X_\bullet) > m$, then $\min(X_\bullet) > m = \max(M_\bullet^Y)$, and so Lemma 2.16 tells us that $\text{Hom}_{D_{\mathcal{L}}^b(\mathcal{T})}(X_\bullet, M_\bullet^Y) = 0$. It follows that $u = 0$. Hence, we have $\delta = \beta^Y \circ u = 0$. So $f = \iota(g) \in D^b(\mathcal{T} \cap \mathcal{L})$. To summarize, we have shown that whenever $\min(X_\bullet) > \min(Y_\bullet) = m$ and $\text{Width}(X_\bullet \oplus Y_\bullet) \leq k$, we have $\text{Hom}_{D^b(\mathcal{T} \cap \mathcal{L})}(X_\bullet, Y_\bullet) \twoheadrightarrow \text{Hom}_{D_{\mathcal{L}}^b(\mathcal{T})}(X_\bullet, Y_\bullet)$.
- Case (ii) If $\min(X_\bullet) = \min(Y_\bullet) = m$, then by Lemma 3.4 $\text{Width}(C_\bullet^X \oplus Y_\bullet) \leq k$ and $\min(C_\bullet^X) \geq m + 1 > m = \min(Y_\bullet)$. Thus, C_\bullet^X, Y_\bullet satisfy the hypothesis of the already proved case(i) and so we have $\text{Hom}_{D^b(\mathcal{T} \cap \mathcal{L})}(C_\bullet^X, Y_\bullet) \twoheadrightarrow \text{Hom}_{D_{\mathcal{L}}^b(\mathcal{T})}(C_\bullet^X, Y_\bullet)$. Hence we have $v \in D^b(\mathcal{T} \cap \mathcal{L})$, and hence $\delta = v \circ \gamma^X \in D^b(\mathcal{T} \cap \mathcal{L})$. Therefore, $f = \iota(g) + \delta$ is in $D^b(\mathcal{T} \cap \mathcal{L})$.
- Case (iii) If $\min(X_\bullet) = m < \min(Y_\bullet)$, then $\text{Width}(C_\bullet^X \oplus Y_\bullet) \leq k - 1$ by Lemma 3.4. Therefore $v \in D^b(\mathcal{T} \cap \mathcal{L})$ by the induction hypothesis, and, as in case(ii), $f \in D^b(\mathcal{T} \cap \mathcal{L})$.

This finishes the proof. \square

Theorem 4.5. *The functor $\iota : D^b(\mathcal{T} \cap \mathcal{L}) \rightarrow D_{\mathcal{L}}^b(\mathcal{T})$ is an equivalence.*

Proof. By Proposition 4.4, ι is full and essentially surjective. Furthermore, ι preserves homologies, thus $\iota(X_\bullet)$ is acyclic if and only if X_\bullet is acyclic. Therefore, ι is faithful on objects. The equivalence then follows from Lemma 2.17. \square

We now come to the case when $\mathcal{T} = \overline{\mathcal{A}}$ where \mathcal{A} is a resolving subcategory of $\text{mod}(R)$.

Lemma 4.6. *There is an equivalence of categories induced by chain complex functors $\iota' : D_{\mathcal{L}}^b(\mathcal{A}) \xrightarrow{\sim} D_{\mathcal{L}}^b(\overline{\mathcal{A}})$.*

Proof. The proof of essential surjectivity is similar to the proof in Lemma 2.15(3). The proof that ι' is fully faithful is easy to obtain and standard. \square

As a straightforward consequence of Lemma 4.6 and Theorem 4.5 when $\mathcal{T} = \overline{\mathcal{A}}$, we can now obtain the main result.

Theorem 4.7. *There is an equivalence of categories $D^b(\overline{\mathcal{A}} \cap \mathcal{L}) \simeq D_{\mathcal{L}}^b(\mathcal{A})$.*

The following interesting corollary can now be obtained from the above result.

Corollary 4.8. *Suppose R is coequidimensional and Cohen-Macaulay, then we have the equivalence*

$$D^b(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{fl}) \simeq D_{fl}^b(\mathcal{P}(R)).$$

Proof. When R is Cohen-Macaulay and coequidimensional, the category $\mathcal{L} = \mathcal{M}_{fl}$ is a Serre subcategory satisfying condition (*). Hence, the equivalence follows from Theorem 4.7 by taking $\mathcal{L} = \mathcal{M}_{fl}$ and $\mathcal{A} = \mathcal{P}(R)$. \square

Furthermore, our result holds for any resolving subcategory in Example 2.3 and Serre subcategory in Example 2.10. We explicitly state some important cases. In the special case when R is Cohen-Macaulay and equicodimensional, $\mathcal{A} = \text{mod}(R)$ and $\mathcal{L} = \mathcal{M}_{fl}$, we obtain the well-known equivalence used in most dévissage statements (refer [12, 1.15(Lemma, Ex. (b))]).

Corollary 4.9.

$$D^b(\mathcal{M}_{fl}) \simeq D_{fl}^b(\text{mod}(R)).$$

Note that this equivalence is known even without the assumption that R is Cohen-Macaulay.

Corollary 4.10. *Let R be Cohen-Macaulay. Let V be any set theoretic complete intersection in $\text{Spec}(R)$ and c be any integer. Let $\overline{\mathcal{P}(R)}_V^c$ denote the category of modules with finite projective dimension supported on V and in codimension at least c . Let $D_V^c(\mathcal{P}(R))$ denote the derived category with chain complexes of projective modules with homologies supported on V and in codimension at least c . Then*

$$D^b(\overline{\mathcal{P}(R)}_V^c) \simeq D_V^c(\mathcal{P}(R)).$$

Note that without c , the above result holds even without R being Cohen-Macaulay.

The main theorem, Theorem 4.7 is also related to an interesting corollary of the oft-quoted Hopkins-Neeman theorem [11, 15] for perfect complexes. Let \mathcal{L} be any Serre subcategory of $\text{mod}(R)$. A consequence of the Hopkins-Neeman theorem is that $\text{thick}_{D^b(\text{mod}(R))}(\overline{\mathcal{P}(R)} \cap \mathcal{L}) \simeq D_{\mathcal{L}}^b(\mathcal{P}(R))$ where *thick* is the thick closure (note that here we use thick in the triangulated sense). We generalize this as follows.

Corollary 4.11. *Let \mathcal{L} be a Serre subcategory satisfying condition (*). Let \mathcal{T} be any thick subcategory of $\text{mod}(R)$. Then the thick closure (in the triangulated sense) of $\mathcal{T} \cap \mathcal{L}$ in $D^b(\text{mod}(R))$ is $D_{\mathcal{L}}^b(\mathcal{T})$ (after completion with respect to isomorphisms).*

Proof. Note that there is a commutative square

$$\begin{array}{ccc} K_{\mathcal{T}}^+(\mathcal{P}(R)) & \longrightarrow & K^+(\mathcal{P}(R)) \\ \downarrow \wr & & \downarrow \wr \\ D^+(\mathcal{T}) & \longrightarrow & D^+(\text{mod}(R)). \end{array}$$

The top horizontal arrow is a full embedding, hence so is the bottom. Hence, restricting to the bounded category, $D^b(\mathcal{T})$ is a thick subcategory of $D^b(\text{mod}(R))$ and hence so is $D_{\mathcal{L}}^b(\mathcal{T})$ (after completing them with respect to isomorphisms). However, up to completion with respect to isomorphisms,

$$\text{image}(D^b(\mathcal{T} \cap \mathcal{L})) \subseteq \text{thick}_{D^b(\text{mod}(R))}(\mathcal{T} \cap \mathcal{L}) \subseteq D_{\mathcal{L}}^b(\mathcal{T}).$$

But by Theorem 4.5, we then get the required result. \square

Finally we prove the theorem we promised at the beginning of the article.

Theorem 4.12. *Suppose R is local. Then R is Cohen-Macaulay if and only if the following equivalence holds*

$$D^b(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{fl}) \simeq D_{fl}^b(\mathcal{P}(R)).$$

Proof. When R is Cohen-Macaulay, the equivalence follows from Theorem 4.7 by taking $\mathcal{L} = \mathcal{M}_{fl}$ and $\mathcal{A} = \mathcal{P}(R)$.

Now suppose R is not Cohen-Macaulay. The new intersection theorem, [20], asserts that such a ring R never admits a finite length, finite projective dimension module. Thus $D^b(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{fl}) = 0$. However, the Hopkins-Neeman theorem [11, 15] states that the thick subcategories of $D^b(\mathcal{P}(R))$ are in bijective correspondence with specialization closed subsets of $\text{Spec}(R)$ and that this bijection is given by taking the supports of objects in $D^b(\mathcal{P}(R))$. Letting m be the maximal ideal of R , since the specialization closed set $\{m\}$ differs from the specialization closed subset \emptyset , the Hopkins-Neeman theorem implies that the subcategory of $D^b(\mathcal{P}(R))$ supported on $\{m\}$, which is $D_{fl}^b(\overline{\mathcal{P}(R)})$, differs from that supported on \emptyset , which is 0. Hence $D_{fl}^b(\overline{\mathcal{P}(R)})$ cannot be 0 and so the equivalence fails. \square

5. RESULTS ON HOMOLOGICAL FUNCTORS

Now that we have proved the equivalence of the two categories, we can compare various theories of invariants for derived equivalences. We restrict our attention to \mathbb{K} -theory and triangular Witt and Grothendieck-Witt groups.

\mathbb{K} -THEORY COMPARISONS AND RESULTS

Since K -theoretic invariants need not always be preserved by equivalences of derived categories ([23]), we will need to view the categories above with some more structure. While the original and several other articles ([30],[28],[16],[2],[3],[31],[24],[25]) serve as good references for this part, we will refer to the articles ([26],[29]) for the terminology and results.

The categories $\text{Ch}_{\mathcal{L}}^b(\mathcal{A})$, $\text{Ch}^b(\mathcal{T} \cap \mathcal{L})$ and $\text{Ch}_{\mathcal{L}}^b(\mathcal{T})$ are strongly pretriangulated dg -categories and the natural functors are functors of such categories. In particular, with the usual choices of weak equivalences as quasi-isomorphisms, these are all complicit exact categories with weak equivalences, and the natural functors preserve weak equivalences. Assume, as usual, that \mathcal{L} also satisfies condition (*).

Let \mathbb{K} be the nonconnective K -theory spectrum. Applying Theorem 4.5 and [26, Theorem 3.2.29], we get that ι induces homotopy equivalences of \mathbb{K} -theory spectra. Similarly, applying Lemma 4.6 and [26, Theorem 3.2.29], we get that ι' induces homotopy equivalences of \mathbb{K} -theory spectra.

Putting these together and further using [26, 3.2.30], we obtain

Theorem 5.1. *The spectra $\mathbb{K}(\overline{\mathcal{A}} \cap \mathcal{L})$ and $\mathbb{K}(D_{\mathcal{L}}^b(\mathcal{A}))$ are homotopy equivalent. Hence*

$$\mathbb{K}_i(\overline{\mathcal{A}} \cap \mathcal{L}) \simeq \mathbb{K}_i(D_{\mathcal{L}}^b(\mathcal{A})) \quad \forall i \in \mathbb{Z}.$$

Once again, this result holds for any resolving subcategory in Example 2.3 and Serre subcategory in Example 2.10. We list the most important corollary.

Corollary 5.2. *Let R be Cohen-Macaulay and equidimensional. Then there is a homotopy equivalence between $\mathbb{K}(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{fl})$ and $\mathbb{K}(D_{fl}^b(\mathcal{P}(R)))$. Hence*

$$\mathbb{K}_i(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{fl}) \simeq \mathbb{K}_i(D_{fl}^b(\mathcal{P}(R)))$$

As mentioned in Section 1, special cases of Corollary 5.2 were known earlier.

When R is Cohen-Macaulay and equicodimensional of dimension d , the special case of $\mathcal{A} = \text{mod}(R)$ and $c = d$ in Theorem 5.1 gives us the well-known equivalence for coherent \mathbb{K} -theory.

Remark 5.3. Let X be a (topologically) noetherian scheme with a bounded generalized dimension function as in [4]. Then coniveau and niveau spectral sequences are defined in [4, Theorem 1, Theorem 2] converging to the \mathbb{K} -groups of X . The q^{th} row on the E_1 page consists of unaugmented Gersten-like complexes with terms

$$\bigoplus_{x \in X^{(p)}} \mathbb{K}_{-p-q}(\mathcal{O}_{X,x} \text{ on } x) \quad \text{and} \quad \bigoplus_{x \in X_{(-p)}} \mathbb{K}_{-p-q}(\mathcal{O}_{X,x} \text{ on } x)$$

respectively. Further, there is an augmented weak Gersten complex for the usual codimension and dimension functions, as defined in [4]. When X is regular, Quillen's dévissage theorem can be applied to rewrite these terms as the \mathbb{K} -theories of the residue fields of the points. However, when X is not regular, the theorem does not apply and hence the terms remain as abstract \mathbb{K} -groups of derived categories supported at the points.

Now under the further assumption that all the local rings $\mathcal{O}_{X,x}$ are Cohen-Macaulay (i.e. X is Cohen-Macaulay), we can apply Theorem 5.2 and rewrite these spectral sequences in terms of the \mathbb{K} -groups of the category of finite length, finite projective dimension modules over the local rings at the points. Thus the computation of global \mathbb{K} -groups can now be reduced to computing \mathbb{K} -groups of the category $\mathcal{P}(R) \cap \mathcal{M}_{fl}(R)$ for a Cohen-Macaulay local ring R .

To summarize, we obtain :

Theorem 5.4. *For a Cohen-Macaulay scheme X of dimension d , we have spectral sequences :*

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} \mathbb{K}_{-p-q}(\overline{\mathcal{P}(\mathcal{O}_{X,x})} \cap \mathcal{M}_{fl}(\mathcal{O}_{X,x})) \xrightarrow{p+q=n} \mathbb{K}_{-n}(X) \quad \text{and} \\ E_1^{p,q} = \bigoplus_{x \in X_{(p)}} \mathbb{K}_{-p-q}(\overline{\mathcal{P}(\mathcal{O}_{X,x})} \cap \mathcal{M}_{fl}(\mathcal{O}_{X,x})) \xrightarrow{p+q=n} \mathbb{K}_{-n}(X)$$

and augmented weak Gersten complexes for each $q \in \mathbb{Z}$:

$$\mathbb{K}_q(X) \rightarrow \bigoplus_{x \in X^{(0)}} \mathbb{K}_q(\overline{\mathcal{P}(\mathcal{O}_{X,x})} \cap \mathcal{M}_{fl}(\mathcal{O}_{X,x})) \rightarrow \\ \bigoplus_{x \in X^{(1)}} \mathbb{K}_{q-1}(\overline{\mathcal{P}(\mathcal{O}_{X,x})} \cap \mathcal{M}_{fl}(\mathcal{O}_{X,x})) \rightarrow \dots \rightarrow \bigoplus_{x \in X^{(d)}} \mathbb{K}_{q-d}(\overline{\mathcal{P}(\mathcal{O}_{X,x})} \cap \mathcal{M}_{fl}(\mathcal{O}_{X,x})).$$

WITT AND GROTHENDIECK-WITT GROUP COMPARISONS AND RESULTS

Let R be an equicodimensional Cohen-Macaulay ring of dimension d . We now consider the situation where the category \mathcal{A} is a duality-closed thick subcategory of \mathcal{G}_C , the category arising from a semidualizing module C as defined in Section 2 with duality given by $\text{Hom}(_, C)$. Let $\mathcal{L} = \mathcal{M}_{fl}$. We assume further that 2 is invertible in the ring R .

There is a duality on the category $\overline{\mathcal{A}} \cap \mathcal{L}$ given by $\text{Ext}_R^d(_, C)$ which induces a duality on $D^b(\overline{\mathcal{A}} \cap \mathcal{L})$. Similarly, there is a duality on $D_{\mathcal{L}}^b(\mathcal{A})$ given by $\text{Hom}_R(_, C)$ (or \dagger) respectively.

The arguments in either of [6, Lemma 6.4], [10] or [13] go through with minimal modifications showing that these are indeed dualities and that they are preserved by the "resolution functor", the composite functor $\iota \circ \iota'^{-1}$ from $D^b(\bar{\mathcal{A}} \cap \mathcal{L})$ to $D_{\mathcal{L}}^b(\mathcal{A})$.

A direct application of [6, Lemma 4.1(c)] now yields

Theorem 5.5. *There is an isomorphism of triangular Witt groups*

$$W(\bar{\mathcal{A}} \cap \mathcal{L}) \xrightarrow{\sim} W^0(D^b(\bar{\mathcal{A}} \cap \mathcal{L})) \xrightarrow{\sim} W^d(D_{\mathcal{L}}^b(\mathcal{A})).$$

The case $\mathcal{A} = \mathcal{P}(R)$ results in the following new corollary.

Corollary 5.6. *When R is equicodimensional, there is an isomorphism of triangular Witt groups*

$$W(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{fl}) \xrightarrow{\sim} W^d(D_{fl}^b(\mathcal{P}(R)))$$

The special case of R being equicodimensional and $C = D$ a dualizing module (or complex) gives us the result in the well-known coherent case [10].

Remark 5.7. Similarly, a direct application of [31, Theorem 2.1] yields the same results for Grothendieck-Witt groups as for Witt groups above.

In [13], the authors define a new Witt group for exact subcategories of triangulated categories closed under duality. They further prove another form of dévissage for triangular Witt groups, namely

$$\begin{array}{ccc} W(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{fl}) & \xrightarrow{\sim_{\alpha}} & W(D_{\overline{\mathcal{P}(R)} \cap \mathcal{M}_{fl}}^b(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{fl})) & \xrightarrow{\sim_{\alpha'}} & W(D^b(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{fl})) \\ & & \downarrow \beta \wr & & \\ & & W^d(D_{fl}^b(\mathcal{P}(R))) & & \end{array}$$

With our notations as above, we can now generalize and improve upon this picture to obtain the following theorem.

Theorem 5.8. *There are natural isomorphisms of Witt groups :*

$$\begin{array}{ccccc} W(\bar{\mathcal{A}} \cap \mathcal{L}) & \xrightarrow{\sim_{\alpha}} & W(D_{\bar{\mathcal{A}} \cap \mathcal{L}}^b(\bar{\mathcal{A}} \cap \mathcal{L})) & \xrightarrow{\sim_{\alpha'}} & W(D_{\mathcal{L}}^b(\bar{\mathcal{A}} \cap \mathcal{L})) \\ & & \downarrow \beta \wr & & \downarrow \gamma \wr \\ & & W^d(D_{\bar{\mathcal{A}} \cap \mathcal{L}}^b(\mathcal{A})) & \xrightarrow{\sim_{\delta}} & W^d(D_{\mathcal{L}}^b(\mathcal{A})) \end{array}$$

Proof. We only give a sketch of the proof. The isomorphism γ already occurs in Theorem 5.5 and β is also a similar direct consequence of Theorem 4.7 by restricting the equivalence of categories to the categories with support $\bar{\mathcal{A}} \cap \mathcal{L}$ (or by following the arguments in [13]). The arguments in [13] generalize directly to give the isomorphisms α and α' . The commutativity of the diagram shows that δ is also an isomorphism. \square

Remark 5.9. The remark 5.3 and succeeding theorem 5.4 works (as in [4, Remark 3]) for any cohomology theory which induces long exact sequences on short exact sequences of triangulated categories. In particular, for Witt theory, we would get spectral sequences and augmented weak Gersten-Witt complexes for triangular Witt groups tensored with $\mathbb{Z}[\frac{1}{2}]$ as in Theorem 5.4.

6. EXAMPLES AND QUESTIONS

An advantage of working with arbitrary Serre subcategories is the ability to deal with supports. We are able to deal with supports only when the Serre subcategory satisfies the condition (*). In contrast, in the coherent picture (i.e. G theory or coherent Witt groups), theorems similar to the ones in section 5 exist with arbitrary supports, i.e. supports in any specialization closed subset, in particular over any closed subset V of $\text{Spec}(R)$. This is one reason why smoothness has played a crucial role in results for \mathbb{K} -theory or other similar theories, since both coherent and usual theories coincide. Thus the following question is natural :

Question 1. *Is $D^b(\mathcal{T} \cap \mathcal{L}) \rightarrow D_{\mathcal{L}}^b(\mathcal{T})$ an equivalence for any thick subcategory \mathcal{T} of $\text{mod}(R)$ and any Serre subcategory \mathcal{L} of $\text{mod}(R)$?*

As we noted in Section 1, when R is local and not Cohen-Macaulay, this is always false with $\mathcal{L} = \mathcal{M}_{fl}$ and $\mathcal{T} = \overline{\mathcal{P}(R)}$. However, it is still plausible that the result holds for a more general class of Serre subcategories.

Next, we consider a question about quotients. Supports allow one to write localization exact sequences. Comparing supports with quotients is a powerful tool, for example it is known that $\mathbb{K}_i\left(\text{mod}\left(\frac{R}{(a)}\right)\right) \simeq \mathbb{K}_i(\{\text{modules supported on } V(a)\})$ where a is a nonzero divisor. This is because one can either apply dévissage directly, or, for other similar theories, the spectral sequence and dévissage reduces one to the case of residue fields of points and both sides have the same residue fields. This leads to the following question :

Question 2. *Let \mathcal{L} be the Serre subcategory of modules supported on $V(I)$ where I is a set theoretic complete intersection ideal. Let \mathcal{T} be a thick subcategory in $\text{mod}(R)$. Is $\mathbb{K}_i\left(D^b\left(\mathcal{T} \cap \text{mod}\left(\frac{R}{I}\right)\right)\right) \rightarrow \mathbb{K}_i\left(D_{\mathcal{L}}^b(\mathcal{T})\right)$ an isomorphism?*

We present a rather simple example which answers this question negatively.

Example 6.1. Let $R = \frac{k[X]}{(X^2)}$. Let $I = (X)$, so $V(I) = \text{Spec}(R)$. Note that $V(I) = V(\emptyset)$ and so $V(I)$ is a set theoretic complete intersection. Let $\mathcal{T} = \overline{\mathcal{P}(R)}$. Then $\mathcal{T} \cap \text{mod}(R/I) = \{0\}$ while $D_{\mathcal{L}}^b(\mathcal{T}) = D^b(\overline{\mathcal{P}(R)}) \neq \{0\}$. Then $K_0(\mathcal{T} \cap \text{mod}(R/I)) = 0$ and $K_0(D_{\mathcal{L}}^b(\mathcal{T})) = \mathbb{Z}$.

Clearly going modulo any ideal I does not work. We specialize to the case when $I = (a)$ where a is a nonzero divisor.

Question 3. *Let a be a nonzero divisor. Let \mathcal{L} be the Serre subcategory of modules supported on $V(a)$. Is $\mathbb{K}_i\left(\overline{\mathcal{A}} \cap \text{mod}\left(\frac{R}{(a)}\right)\right) \rightarrow \mathbb{K}_i(\overline{\mathcal{A}} \cap \mathcal{L})$ an isomorphism?*

One natural way to answer this question would involve the following two steps :

- (1) Reprove Quillen's dévissage theorem for full subcategories of $\text{mod}(R)$ satisfying the 2-out-of-3 property (it is known for abelian subcategories).
- (2) Find a natural filtration for a module $M \in \overline{\mathcal{A}} \cap \mathcal{L}$ so that the quotients belong to $\overline{\mathcal{A}} \cap \text{mod}\left(\frac{R}{(a)}\right)$.

For every module M on the right there exists an n such that $a^n M = 0$ and thus a natural filtration : $M \supseteq aM \supseteq \dots \supseteq a^{n-1}M \supseteq 0$ and $\frac{a^i M}{a^{i+1}M}$ is in $\text{mod}\left(\frac{R}{(a)}\right)$. At

first sight, this might seem like an answer to the second part. However, it turns out that even though $M \in \mathcal{A}$, its quotient $\frac{M}{aM}$ need not be. Hailong Dao provided us with the following example in the best possible case of a polynomial variable.

Example 6.2. Let $R = \frac{k[[X, Y]]}{(XY)}[Z]$. Let $M = \frac{R}{(X - Z, Y - Z)}$. Then M has finite projective dimension over R , $Z^2M = 0$ but $\frac{M}{(Z)M} = k$ does not have finite projective dimension over $\frac{k[[X, Y]]}{(XY)}$.

Since the module M has length 2, there is no other option of a filtration. Thus, the most natural naive arguments provide no answer. If Question 3 has a positive answer, it would yield nice long exact sequences and be useful in computations.

Finally, let R be equidimensional and Cohen-Macaulay of dimension d , $\mathcal{A} \subseteq \mathcal{G}_C$ a thick subcategory closed under the duality, and \mathcal{L} the Serre subcategory of finite length modules supported on a set theoretic complete intersection V . Since we have an equivalence of triangular Witt groups and triangular Grothendieck-Witt groups (Theorem 5.5, Remark 5.7) and both are obtained from a Grothendieck-Witt spectrum, it begs the natural question :

Question 4. *Are the Grothendieck-Witt spectra of $D^b(\overline{\mathcal{A}} \cap \mathcal{L})$ and $D^b_{\mathcal{L}}(\mathcal{A})$ homotopy equivalent?*

If a suitable intermediate category can be found which has duality, then we can answer this in the affirmative. However, there is in general no duality on $\overline{\mathcal{A}}$. Also, we do not know if the category of double complexes constructed in the proof of [6, Lemma 6.4] arises as the homotopy category of some suitable pretriangulated category.

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